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Berliant, M.; Page Jr., F.H.

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By Marcus Berliant and Frank H. Page, Jr.

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by

Marcus Berliant
Department of Economics
Washington University
Campus Box 1208
One Brookings Drive
St. Louis, MO 63130-4899
Tele: (314) 935-5691
FAX: (314) 935-4156
e-mail: berliant@wuecona.wustl.edu

and

Frank H. Page, Jr.*
Economics, Finance, and Legal Studies
University of Alabama
Tuscaloosa, AL 35487
Tele: (205) 348-6097
FAX: (205) 348-0590
e-mail: fpage@alston.cba.ua.edu

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Abstract

In this paper we establish necessary and sufficient conditions for the simultaneous existence of an optimal income tax mechanism and an optimal vector of public goods. Moreover, we identify a condition sufficient to guarantee that the optimal mechanism is budget balancing. The key ingredient in our analysis is a result characterizing incentive compatible income tax/public goods mechanisms. This result allows us to convert the tax design/public goods problem with financing and incentive compatibility constraints to an equivalent design problem without incentive compatibility constraints. Our characterization of incentive compatibility requires only very weak assumptions concerning agents' utility functions and does not rely in any way on the problematic first order approach. Thus, gaps and bunching are permitted. While much of the literature restricts optimal taxes to be in certain classes of functions, our only restriction on the class of income tax functions is measurability.

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1. Introduction

Since the seminal work of Mirrlees (1971), economists have used models of optimal income taxation for policy prescriptions as well as normative analysis of models of government behavior. Although it is usually necessary to employ simulations since closed form solutions to the optimal tax problem are often unavailable, the model seems capable of yielding important insights into tax design since it combines government optimization with individual behavior in the context of uncertainty about the types or wage rates of agents. The incentive compatibility constraints on the government that arise naturally from this uncertainty place interesting and vital limits on government behavior. Examples of the model's usefulness include Brunner (1989), Tillman (1989), Tuomala (1990), and Weymark (1986a and b, 1987).

The model also suffers from some notable defects. First, it is generally difficult to give necessary (or sufficient) conditions for an optimal income tax other than the standard condition that the top ability individual(s) face a marginal tax rate of zero. Further properties of an optimal income tax are derived only from simulation. Second, it is convenient to replace the optimization problem of agents with the associated first order conditions for optimization (the so-called first order approach to incentive compatibility) both for analytical tractability and for simulations. Unfortunately, as L'Ollivier and Rochet (1983) show using an example, some optimal taxes involve bunching (having multiple types earning the same gross income) or gaps (having no types earning some incomes), which implies that the first order approach is not valid in the sense that a true optimal tax might not satisfy the first order conditions. It also implies that income taxes derived using the first order approach are not necessarily optimal, as they might violate second order conditions for the consumer optimization problems. Berliant and Gouveia (1994) find conditions on primitives of the optimal income tax problem sufficient to obtain validity of the first order approach to incentive compatibility, but these conditions are rather stringent, as they involve additive separability of consumer utility functions and conditions on the third derivatives.

Further problems with the model include the restriction to one-dimensional type descriptions (agents are differentiated only by the wage rate) and strong assumptions concerning the properties of utility functions, generally including normality of one or both goods, a single crossing property, smoothness, and quasi-concavity.

In this paper we establish necessary and sufficient conditions for the simultaneous existence of an optimal income tax mechanism and an optimal vector of public goods. More importantly, we identify a condition sufficient to guarantee that an optimal tax mechanism can be chosen so to generate the *exact* amount of revenue required to finance the optimal vector of public goods. Thus an optimal tax mechanism can be chosen that is budget balancing.

Our analysis is carried out in a general setting, independent of the validity of the first order approach to incentive compatibility, and requires only very weak assumptions on consumer utility functions.[1] No single crossing property is assumed, and utility functions need not even be quasi-concave or have any normality property. Gaps and bunching are permitted. The techniques employed are sufficiently general to allow for multidimensional (and even infinite dimensional) agent type descriptions. Quinzii and Rochet (1985) found the first order approach to such models to be exceedingly messy.

The model developed here, while similar to models found in the principal-agent literature (e.g., Mirrlees (1976), Holmstrom (1979)), differs from the standard principal-agent model in several important respects. First, rather than there being a single agent, in our model there are uncountably many agents. Second, in our model agents face no uncertainty once they have chosen an action. In particular, each agent chooses a level of income rather than a probability distribution over income. Finally, in our model there are no voluntary participation (or individual rationality) constraints. These constraints are replaced by a financing constraint which requires that the government choose a tax mechanism that finances the public goods.

Because public goods are financed from current consumption via the income tax, the government in choosing a vector of public goods and a tax function must be concerned with the incentives for subsequent income generation their choices create. In analyzing the government's tax design/public goods problem we explicitly take into account these incentives. Thus we formally examine the trade-off between the welfare enhancing effects of public goods versus the adverse incentives effects of taxation.

[1]Using variational techniques, Brito and Oakland (1977) give necessary conditions the optimal quantity of public goods will satisfy if financed by an optimal income tax. Besides carrying out our analysis in a more general setting, our focus here is upon the simultaneous existence of an optimal tax mechanism and an optimal quantity of public goods.

Much of the tax literature simply restricts optimal taxes to be in certain classes of functions (e.g., a class of equicontinuous functions) to obtain the existence of an optimum. Of course once this restriction is made, it is possible that an income tax function not in this class dominates the optimum in this class. For instance, if an optimal income tax is found in the class of differentiable functions, it is possible that an income tax in the class of continuous functions dominates it. If an optimal tax is found in the class of continuous functions, it is possible that an income tax in the class of piecewise continuous functions dominates it, and so forth. In the analysis below no substantial restrictions are placed upon the class of income tax functions considered. Thus, the optimal income tax function is determined by economic considerations rather than exogenous technical restrictions.

In the work presented here, we find necessary and sufficient conditions for the simultaneous existence of an optimal income tax mechanism and an optimal vector of public goods. The modeling assumptions required for these conditions to be valid are surprisingly weak - the most critical assumption being the existence of a direct tax function and a vector of public goods satisfying the financing constraint (i.e., the requirement that the income tax function generate enough revenue to finance the vector of public goods). This assumption is similar to the Slater condition in the context of mathematical programming and can be quite easily checked in many problems.

The existence question centers on whether or not the constrained mathematical programming problem describing the tax design/public goods problem has a solution. Because there can be uncountably many consumer types (i.e., wage rates), the tax design problem can have uncountably many incentive compatibility constraints. This, of course, greatly complicates the existence problem. The key ingredient in our analysis is a result characterizing (multi-dimensional and even infinite dimensional) incentive compatibility that allows us to *convert the tax design/public goods problem with financing and incentive compatibility constraints to an equivalent design problem without incentive compatibility constraints*. The existence of an optimal income tax mechanism and an optimal vector of public goods can then be established within a very general class of models using only classical results (e.g., a continuous function on a compact set achieves a maximum).

Before proceeding with the analysis, two remarks are in order. First, while we focus on existence, we believe that the techniques developed here will be useful in analyzing the properties of optimal tax mechanisms and optimal levels of public goods. Second, Kaneko (1981) proves existence of an optimal tax in a different but related model.

In Section 2 we present the basic ingredients of the model, state the mechanism design problem corresponding to the optimal tax/public goods problem, and discuss efficiency. In Section 3 we discuss income/tax menus, financing requirements, and incentive compatibility. Moreover, in Section 3 we present our characterization of incentive compatible public sector mechanisms. In Section 4 we establish necessary and sufficient conditions for the simultaneous existence of an optimal income tax mechanism and an optimal vector of public goods. Finally, in Section 5 we identify a condition sufficient to guarantee the existence of an optimal tax mechanism that generates the exact amount of revenue required to finance the optimal vector of public goods.

2. The Framework

Basic Ingredients

Let Y and T denote two closed bounded intervals of R_+ (the nonnegative real numbers) such that Y=T. In particular, let $Y = T = [0, m]$ for some large positive real number m. Consider the set

$$K = \{(y, \tau) \in Y \times T : y \geq \tau\}. \tag{1}$$

K is the set of all *feasible* income and tax liability pairs in $Y \times T$. Equipped with the standard Euclidean metric, $d_e(\cdot,\cdot)$, K is compact.

Now let G be a compact subset of R_+^k, and let $z = (z_1, \ldots, z_k)$ denote a typical element in G. Each vector z is a vector of public goods. For each vector z of public goods, let c(z) denote the (nonnegative) cost of providing public goods z. In the model we develop here, the cost public goods will be financed from consumption via the tax mechanism.

Denote by W the set of agent types, usually called ability or wage rates in the literature, and equip W with a σ – field Σ and a probability measure $P(\cdot)$ defined on Σ. For $E \in \Sigma$, P(E) is the fraction of the total number of agents that are of type $w \in E$.

Finally, for each agent type $w \in W$, let $u(w,\cdot,\cdot,\cdot) : K \times G \to R$ denote the agent's utility function defined over 3-tuples of income, tax liability, and public goods, $(y, \tau, z) \in K \times G$. We will assume the following concerning agents' utility functions:

[A-1]: (1) For each $w \in W$, $u(w,\cdot,\cdot,\cdot)$ is continuous on $K \times G$, and for each $(y, \tau, z) \in K \times G$, $u(\cdot, y, \tau, z)$ is Σ – measurable.

(2) For each $(w, y, z) \in W \times Y \times G$, $u(w, y, \cdot, z)$ is strictly decreasing on $K(y) = \{\tau : (y, \tau) \in K\}$ (i.e., if τ and τ' are in K(y) and $\tau < \tau'$, then $u(w, y, \tau', z) < u(w, y, \tau, z)$).

EXAMPLE 1:

Suppose agents have preferences defined over nonnegative values for consumption c, labor ℓ, and public goods z represented by a continuous utility function, $v(\ell,c,z)$ which is strictly increasing in consumption. Suppose also that agents differ by an ability parameter, w, strictly positive which can be interpreted as a wage rate or productivity. In particular, let $W = [L,H] \subset R_{++}$ denote the set of all possible ability parameters and equip W with the Borel σ-field. Finally, suppose that for each income and tax liability pair $(y,\tau) \in K$, labor is given by $\ell = \frac{y}{w}$, and consumption by $c = y - \tau$. The utility function $u(\cdot,\cdot,\cdot,\cdot)$ given by $u(w,y,\tau,z) = v(\frac{y}{w}, y-\tau, z)$ satisfies [A-1](1) and (2).

We will also assume the following concerning the cost of providing public goods:

[A-2] The cost function $c(\cdot): G \to R_+$ is lower semicontinuous.[2]

The Tax Design Problem with Public Goods

As in Berliant and Gouveia (1994), we assume that the government does not know each agent's type but can observe each agent's income and thus deduce (the resulting) tax liability.

To begin, let $\mu(\cdot)$ be a countably additive finite measure defined on the measurable space of agent types (W,Σ), equivalent to the probability measure $P(\cdot)$.[3] The measure $\mu(\cdot)$ represents one possible welfare weighting scheme for agent types.

Now let $M(W,Y)$ denote the set of all $(\Sigma, B(Y))$-measurable functions $y(\cdot): W \to Y$, $M(Y,T)$ the set of all $(B(Y),B(T))$-measurable functions $t(\cdot): Y \to T$, and $M(W,G)$ the set of all $(\Sigma, B(G))$-measurable functions $z(\cdot): W \to G$.[4] The μ-tax design problem with public goods is stated as follows:

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[2] $c(\cdot): G \to R_+$ is lower semicontinuous if $z_n \to z$ implies $\liminf_n c(z_n) \geq c(z)$.

[3] μ and P are equivalent if they have the same sets of measure zero. Thus μ and P are equivalent if μ is absolutely continuous with respect to P and P is absolutely continuous with respect to μ.

[4] Here, B(Y) denotes the Borel σ-field in Y, B(T) the Borel σ-field in T, and B(G) the Borel σ-field in G. A function $y(\cdot): W \to Y$ is $(\Sigma, B(Y))$-measurable iff $\{w \in W : y(w) \in E\} \in \Sigma$ for $E \in B(Y)$. $(B(Y), B(T))$-measurability and $(\Sigma, B(G))$-measurability are defined in a similar manner.

$$\text{maximize} \int_W u(w, y(w), t(y(w)), z(w)) d\mu(w) \tag{2}$$

subject to the constraints

$$(y(\cdot), t(\cdot), z(\cdot)) \in M(W,Y) \times M(Y,T) \times M(W,G), \tag{3}$$

$$\text{the function } z(\cdot) \text{ is everywhere constant and equal to some } z \in G, \tag{4}$$

$$\text{for each } w \in W, \quad u(w, y(w), t(y(w)), z(w)) \geq u(w, y, t(y), z(w')) \quad \text{for all } y \in Y \text{ and } w' \in W, \tag{5}$$

$$0 \leq t(y) \leq y \text{ for all } y \in Y, \tag{6}$$

$$\int_W (t(y(w)) - c(z(w))) dP(w) \geq 0. \tag{7}$$

We will refer to any $y(\cdot) \in M(W,Y)$ as a direct income function (since it is defined on types) and any $t(\cdot) \in M(Y,T)$ as an indirect tax function (since it is defined on income rather than types). We will also refer to any function $z(\cdot) \in M(W,G)$ as a direct public goods function. Since the consumption of public goods must be the same for all agents, the feasible set of direct public goods functions consists of constant functions (as specified in (4)). We will refer to any pair of functions $(y(\cdot), t(\cdot)) \in M(W,Y) \times M(Y,T)$ as an income tax mechanism and to any 3-tuple of functions

$$(y(\cdot), t(\cdot), z(\cdot)) \in M(W,Y) \times M(Y,T) \times M(W,G), \tag{8}$$

as a public sector mechanism.

The constraints given by (5) are the incentive compatibility constraints. Note that there can be uncountably many incentive compatibility constraints. Denote by Ψ the subset of public sector mechanisms $(y(\cdot), t(\cdot), z(\cdot))$ satisfying the incentive compatibility constraints *with* $z(\cdot)$ *a constant function*.

The constraint given by (6) is a feasibility constraint requiring that the indirect tax function be such that for all income levels taxes be nonnegative and not exceed income. Denote by Γ the subset of public sector mechanisms $(y(\cdot), t(\cdot), z(\cdot))$ with $t(\cdot)$ satisfying the feasibility constraint.

The constraint given by (7) is the financing constraint. It requires that any public sector mechanism $(y(\cdot), t(\cdot), z(\cdot))$ be such that the total tax revenues generated by the income tax mechanism $(y(\cdot), t(\cdot))$ be sufficient to cover the cost of providing public goods $z(\cdot)$. Denote by Π the subset of public sector mechanisms $(y(\cdot), t(\cdot), z(\cdot))$ satisfying the financing constraint.

Definition 1
We say that the public sector mechanism $(y(\cdot), t(\cdot), z(\cdot))$ implements the indirect tax function $t(\cdot)$ and finances public goods $z(\cdot)$ if and only if $(y(\cdot), t(\cdot), z(\cdot)) \in \Psi \cap \Gamma \cap \Pi$.

Efficiency

We begin with a definition.

Definition 2
We say that a public sector mechanism $(y(\cdot), t(\cdot), z(\cdot)) \in \Psi \cap \Gamma \cap \Pi$ is efficient if and only if there does not exist another public sector mechanism $(y'(\cdot), t'(\cdot), z'(\cdot)) \in \Psi \cap \Gamma \cap \Pi$ such that

$$u(w, y'(w), t'(y'(w)), z'(w)) \geq u(w, y(w), t(y(w)), z(w)) \text{ a.e.}[P] \tag{9}$$

and

$$u(w, y'(w), t'(y'(w)), z'(w)) \geq u(w, y(w), t(y(w)), z(w)) \text{ for all } w \in E, \tag{10}$$

for some $E \in \Sigma$ with $P(E) > 0$.

The following Proposition gives sufficient conditions for efficiency. The proof is straightforward.

Proposition 1
If the mechanism $(y(\cdot), t(\cdot), z(\cdot)) \in \Psi \cap \Gamma \cap \Pi$ solves the design problem ((2)-(7)) for some finite measure μ equivalent to the probability measure P, then $(y(\cdot), t(\cdot), z(\cdot))$ is efficient.

3. Menus, Mechanisms, and Revenue Requirements

Menus and Direct Public Sector Mechanisms

One way to approach the public sector design problem is to view the problem as an optimal delegation problem (e.g., see Holmstrom (1984) or Page (1992)). Viewing the problem in this way, the government simply chooses a menu of public goods and a menu of income and tax liability pairs from some feasible collection of menus and delegates the choice of public goods consumption and the choice of an income and tax liability pair to the agents. There are two problems that must be overcome, however, in order for the delegation approach to the public sector design problem to be valid. First, a feasible collection of menus must be identified that is consistent with the constraints in the design problem. Second, the menu design problem must be shown to be equivalent to the mechanism design problem. In the analysis to follow we will show that both of these difficulties can be easily overcome.

To begin, let $P_f(K)$ denote the collection of all nonempty closed subsets of K (where as before, K is the set of all feasible income and tax liability pairs in $Y \times T$), and equip $P_f(K)$ with the Hausdorff metric h. To accomplish this, define $d_e(s',C) = \inf_{s \in C} d_\eta(s',s)$ where $s' = (y',\tau')$ and $s = (y,\tau)$ are income/tax payment pairs in K and $C \in P_f(K)$. The Hausdorff metric h is then given by

$$h(A,B) = \max\{\sup_{s \in A} d_\eta(s,B), \sup_{s \in B} d_\eta(s,A)\} \text{ for A, B in } P_f(K) . \tag{11}$$

Since K is a compact metric space, $P_f(K)$ equipped with the Hausdorff metric is also a compact metric space (Berge (1963)).

Convergence in $(P_f(K),h)$ can be characterized as follows. Let $\{C_n\}_n$ be a sequence in $P_f(K)$ and define $Li(C_n)$ as follows: $s \in Li(C_n)$ if and only if there is a sequence $\{s_n\}_n$ in K such that for each n $s_n \in C_n$ and $\lim_n s_n = s$. Now define $Ls(C_n)$ as follows: $s \in Ls(C_n)$ if and only if there is a subsequence $\{s_{n_j}\}_j$ in K such that for each j $s_{n_j} \in C_{n_j}$ and $\lim_j s_{n_j} = s$. A subset of income/tax liability pairs $C \in P_f(K)$ is said to be the limit of $\{C_n\}_n$ if $Li(C_n) = C = Ls(C_n)$. Moreover, $h(C_n,C) \to 0$ (i.e., the sequence $\{C_n\}_n$ converges to $C \in P_f(K)$ under the Hausdorff metric h) if and only if $Li(C_n) = C = Ls(C_n)$.

Since the government cannot control or restrict the agent's income choice, any menu $C \in P_f(K)$ chosen by the government must be such that $\text{proj}_Y(C) = Y$, where $\text{proj}_Y(C)$ denotes the projection of the closed set $C \subset Y \times T$ onto Y. Hence menu choice must be restricted to the set Λ, where

$$\Lambda = \{C \in P_f(K) : \text{proj}_Y(C) = Y\}. \tag{12}$$

The set Λ is nonempty (e.g., take the 45 degree line in the square $Y \times T$) and closed with respect to the Hausdorff metric h (i.e., Λ is h-closed).[5] Thus, (Λ, h) is a compact metric space.

Now let $P_f(G)$ denote the collection of all nonempty closed subsets of $G \subset R_+^k$, the feasible set of public goods vectors, and equip $P_f(G)$ with the Hausdorff metric $\bar{h}$. Since G is compact, $(P_f(G), \bar{h})$ is also a compact metric space. In the case of public goods consumption, the public goods consumption choice for each agent must be the same (see expression (4) in the design problem (2)-(7)). In order to capture this constraint in the menu problem, let S denote the collection of all singleton sets (i.e., $H \in S$ if and only if $H = \{z\}$ for some z in G). The collection of single-element menus S is an $\bar{h}$-closed subset of $P_f(G)$. Thus, $(S, \bar{h})$ too is a compact metric space.

Given a particular pair of menus $(C,H) \in \Lambda \times S$ chosen by the government, the resulting choice problem for agents is given by

$$\max_{(y,\tau,z) \in C \times H} u(w,y,\tau,z). \tag{13}$$

Since $C \times H \subset K \times G$ is compact, for each agent type $w \in W$, this problem has a solution. Let

$$u^\wedge(w,C,H) = \max_{(y,\tau,z) \in C \times H} u(w,y,\tau,z), \tag{14}$$

and

$$\Phi(w,C,H) = \{(y,\tau,z) \in C \times H : u(w,y,\tau,z) \geq u^\wedge(w,C,H)\}. \tag{15}$$

[5]In particular, it is easy to show that if $\{C_n\}_n \subset \Lambda$ converges to $C \in P_f(K)$ under the h metric, then $\text{proj}_Y(C) = Y$.

Given menus $(C,H) \in \Lambda \times S$, $u^\wedge(w,C,H)$ is the optimal level of utility attainable by a type w agent, while $\Phi(w,C,H)$ is the set of income, tax liability, and public goods 3-tuples from which the type w agent must choose in order to attain utility level $u^\wedge(w,C,H)$. Thus, the mapping $w \to \Phi(w,C,H)$ is a best response mapping.

Proposition 2

(1) $u^\wedge(w,\cdot,\cdot)$ is continuous on $\Lambda \times S$ for each $w \in W$ (with respect to the product metric) and $u^\wedge(\cdot,C,H)$ is Σ-measurable on W for each $(C,H) \in \Lambda \times S$.

(2) $\Phi(w,C,H) \subset K \times G$ is nonempty and compact for each $(w,C,H) \in W \times \Lambda \times S$. Moreover, $\Phi(w,\cdot,\cdot)$ is upper semicontinuous on $\Lambda \times S$ for each $w \in W$ (with respect to the product metric) and $\Phi(\cdot,\cdot,\cdot)$ is $\Sigma \times B(\Lambda) \times B(S)$-measurable on $W \times \Lambda \times S$.[6]

Proposition 2 essentially summarizes the contents of Propositions 4.1 and 4.2 in Page (1992).

By the Kuratowski, Ryll-Nardzewski Theorem (see Theorem 5.1 in Himmelberg (1975)), given any pair of menus $(C,H) \in \Lambda \times S$ there exists a $(\Sigma, B(Y) \times B(T) \times B(G))$-measurable function[7] $w \to (y(w), \tau(w), z(w))$ such that

$$(y(w), \tau(w), z(w)) \in \Phi(w,C,H) \text{ for all } w \in W, \tag{16}$$

and thus such that for all $w \in W$,

$$u(w, y(w), \tau(w), z(w)) = u^\wedge(w,C,H) = \max_{(y,\tau,z) \in C \times H} u(w,y,\tau,z). \tag{17}$$

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[6]Here $B(\Lambda)$ denotes the Borel σ-field in the compact metric space (Λ, h) and $B(S)$ the Borel σ-field in the compact metric space $(S, \bar{h})$. $\Phi(\cdot,\cdot,\cdot)$ is $\Sigma \times B(\Lambda) \times B(S)$-measurable iff for each closed subset E of $Y \times T \times G$, $\{(w,C,H) \in W \times \Lambda \times S : \Phi(w,C,H) \cap E \neq \emptyset\} \in \Sigma \times B(\Lambda) \times B(S)$ (see Himmelberg (1975)).

[7]The function $w \to (y(w), \tau(w), z(w))$ is $(\Sigma, B(Y) \times B(T) \times B(G))$-measurable iff $w \to y(w)$ is $(\Sigma, B(Y))$-measurable, $w \to \tau(w)$ is $(\Sigma, B(T))$-measurable, and $w \to z(w)$ is $(\Sigma, B(G))$-measurable (see Dudley (1989)).

In fact, it is easy to show that any 3-tuple of functions

$$(y(\cdot), \tau(\cdot), z(\cdot)) \in M(\Sigma, Y) \times M(\Sigma, T) \times M(\Sigma, G) \tag{18}$$

satisfying (16), satisfies for each w and w' in W the inequality

$$u(w, y(w), \tau(w), z(w)) \geq u(w, y(w'), \tau(w'), z(w')). \tag{19}$$

Thus, any 3-tuple of measurable functions $(y(\cdot), \tau(\cdot), z(\cdot))$ satisfying (16) is an incentive compatible, *direct* public sector mechanism corresponding to the pair of menus $(C,H) \in \Lambda \times S$. Moreover, given any incentive compatible direct public sector mechanism $(y(\cdot), \tau(\cdot), z(\cdot))$ corresponding to the menus $(C,H) \in \Lambda \times S$, we have for each $w \in W$

$$u(w, y(w), \tau(w), z(w)) = u(w, y, \tau, z) \text{ for all } (y, \tau, z) \in \Phi(w, C, H). \tag{20}$$

Finally, given any pair of menus $(C,H) \in \Lambda \times S$ and any 3-tuple of measurable functions $(y(\cdot), \tau(\cdot), z(\cdot))$ satisfying (16) , we have for some z in G $z(w) = z$ for all $w \in W$ (recall that $H = \{z\}$ for some z in G).

We will take as the set of all possible *direct* public sector mechanisms, the set of all 3-tuples

$$(y(\cdot), \tau(\cdot), z(\cdot)) \in M(\Sigma, Y) \times M(\Sigma, T) \times M(\Sigma, G).$$

Moreover, given any direct public sector mechanism $(y(\cdot), \tau(\cdot), z(\cdot))$, we will refer to $y(\cdot)$ as the direct income function (as before), $\tau(\cdot)$ as the direct tax function, and $z(\cdot)$ as the direct public goods function (as before).

Menus and Revenue Requirements

A pair of menus $(C,H) \in \Lambda \times S$ is *revenue feasible* if the set-valued mapping $w \to \Phi(w,C,H)$ has a measurable selection $(y(\cdot), \tau(\cdot), z(\cdot))$ such that

$$\int_W (\tau(w) - c(z(w)))dP(w) \geq 0. \tag{21}$$

Consider the problem

$$\sigma(w,C,H) = \max\{\tau - c(z) : (y,\tau,z) \in \Phi(w,C,H)\}, \tag{22}$$

The quantity $\sigma(w,C,H)$ is the maximum amount of tax surplus obtainable from a type w agent consistent with incentive compatibility given menus $(C,H) \in \Lambda \times S$. Since $\Phi(w,C,H) \subset K \times G$ is nonempty and compact, $\sigma(w,C,H)$ is well-defined for each $(w,C,H) \in W \times \Lambda \times S$. Now consider the real-valued mapping $\Delta(\cdot,\cdot)$ defined on $\Lambda \times S$ and given by

$$\Delta(C,H) = \int_W \sigma(w,C,H)dP(w). \tag{23}$$

Proposition 3

(1) $\sigma(\cdot,\cdot,\cdot)$ is $\Sigma \times B(\Lambda) \times B(S)$-measurable and for each $w \in W$, $\sigma(w,\cdot,\cdot)$ is upper semicontinuous on $\Lambda \times S$. Moreover, for each pair of menus $(C,H) \in \Lambda \times S$, there exists a measurable selection $(y(\cdot),\tau(\cdot),z(\cdot))$ from $\Phi(\cdot,C,H)$ such that $\tau(w) - c(z(w)) = \sigma(w,C,H)$ for all $w \in W$.

(2) The mapping $(C,H) \to \Delta(C,H)$ is upper semicontinuous on $\Lambda \times S$.

PROOF: (1) Noting that the function $(\tau,z) \to \tau - c(z)$ is upper semicontinuous, the first part of (1) follows directly from Proposition 4.3 in Page (1992). The second part follows from the Kuratowski, Ryll-Nardzewski Theorem.
(2) Since $(C,H) \to \sigma(w,C,H)$ is upper semicontinuous on $\Lambda \times S$ for each $w \in W$, it follows from Fatou's Lemma that $(C,H) \to \Delta(C,H)$ is upper semicontinuous on $\Lambda \times S$ (see Dudley (1989)). Q.E.D.

Let

$$R = \{(C,H) \in \Lambda \times S : \Delta(C,H) \geq 0\}. \tag{24}$$

R is the set of all revenue feasible menu pairs. In particular, for $(C,H) \in R$,

$$\int_W \sigma(w,C,H)dP(w) \geq 0,$$

and by part (1) of Proposition 3 there is a measurable selection $(y(\cdot),\tau(\cdot),z(\cdot))$ from $\Phi(\cdot,C,H)$ such that $\tau(w)-c(z(w))=\sigma(w,C,H)$ for all $w\in W$. Thus, for this measurable selection

$$\int_W (\tau(w)-c(z(w)))dP(w)\geq 0.$$

We will assume that

[A-3] $$R\neq\varnothing.$$

Proposition 4

R is a closed subset of the compact metric space $\Lambda\times S$.

PROOF: The result follows directly from the definition of upper semicontinuity and the fact that $(C,H)\to\Delta(C,H)$ is h-upper semicontinuous. Q.E.D.

EXAMPLE 2:

Suppose $Y\times T=[0,5]\times[0,5]$ and $G=[0,2]$. Suppose also that agents' ability parameter, w, is distributed uniformly on the closed interval $W=[4,5]$, and that agents have preferences defined over nonnegative values for consumption c, labor ℓ, and public goods z represented by a continuous utility function, $v_\varepsilon(\cdot,\cdot,\cdot):[0,1]\times[0,5]\times[0,2]\to R$, given by

$$v_\varepsilon(\ell,c,z)=(1-\ell+\varepsilon)\cdot(c+\varepsilon)\cdot(z+1),$$

where $\varepsilon>0$ is a small positive number.[8] Letting $\ell=\frac{y}{w}$ and $c=y-\tau$, we have then $u_\varepsilon(\cdot,\cdot,\cdot,\cdot):[4,5]\times K\times[0,2]\to R$ given by

$$u_\varepsilon(w,y,\tau,z)=(1-\frac{y}{w}+\varepsilon)\cdot(y-\tau+\varepsilon)\cdot(z+1),$$

[8]Note that if $\varepsilon=0$, then agents have Cobb-Douglas utility functions. Unfortunately Cobb-Douglas utility functions violate monotonicity at boundaries. Thus, in our example if $\varepsilon=0$ and $\ell=1$, then utility is no longer increasing in consumption and the example will fail to satisfy [A-1](2). Hence, we have $\varepsilon>0$.

and it is easy to verify that $u_{\varepsilon}(\cdot,\cdot,\cdot,\cdot)$ satisfies [A-1] (1) and (2). Finally, suppose that the cost of public good z is given by $c(z) = z$. Thus, the cost function $c(\cdot)$ satisfies [A-2].

If the government chooses menus $(C,H) \in \Lambda \times S$ given by

$$C = \{(y,\tau) \in K : \tau = \frac{1}{2}y,\ 0 \le y \le 2\},$$

and

$$H = \{1\},$$

then the choice problem for each agent, w, is given by

$$\max_{(y,\tau,z) \in C \times \{1\}} u(w,y,\tau,z).$$

For each agent, w, this problem reduces to

$$\max_{y \in [0,5]} (1 - \frac{y}{w} + \varepsilon) \cdot (\frac{1}{2}y + \varepsilon) \cdot (2).$$

Using elementary calculus, it is easy to show that the mapping $w \to \Phi(w,C,H)$ corresponding to this collection of choice problems has a *unique* measurable selection $(y(\cdot), \tau(\cdot), z(\cdot))$ given for each $w \in W$ by

$$y(w) = \frac{1}{2}(1+\varepsilon)w - \varepsilon$$
$$\tau(w) = \frac{1}{4}(1+\varepsilon)w - \frac{1}{2}\varepsilon$$
$$z(w) = 1.$$

The tax surplus function $w \to \sigma(w,C,H)$ (see expression (22)) is then given by

$$\sigma(w,C,H) = \tau(w) - c(z(w))$$
$$= \frac{1}{4}(1+\varepsilon)w - \frac{1}{2}\varepsilon - 1,$$

and thus we have

$$\Delta(C,H) = \int_W \sigma(w,C,H)dP(w)$$

$$= \int_4^5 (\frac{1}{4}(1+\varepsilon)w - \frac{1}{2}\varepsilon - 1)dw$$

$$= .125 + .625\varepsilon.$$

We can conclude, therefore, that the pair of menus $(C,H) \in \Lambda \times S$ given by

$$C = \{(y,\tau) \in K : \tau = \frac{1}{2}y,\ 0 \le y \le 2\} \text{ and } H = \{1\}$$

is revenue feasible and thus is contained in R.

Menus and Public Sector Mechanisms

Next we have our main result characterizing public sector mechanisms in $\Psi \cap \Gamma \cap \Pi$ in terms of menu pairs in R.

Theorem 1

Suppose [A-1], [A-2], and [A-3] hold.

(1) Given any pair of menus $(C,H) \in R$, there exists a public sector mechanism $(y(\cdot), t(\cdot), z(\cdot))$ in $\Psi \cap \Gamma \cap \Pi$ such that

$$(y(w), t(y(w)), z(w)) \in \Phi(w,C,H) \text{ for all } w \in W.$$

(2) Given any public sector mechanism $(y(\cdot), t(\cdot), z(\cdot))$ in $\Psi \cap \Gamma \cap \Pi$, there exists a pair of menus $(C,H) \in R$ such that

$$(y(w), t(y(w)), z(w)) \in \Phi(w,C,H) \text{ for all } w \in W.$$

PROOF: (1) First, let $(C,H) \in R$ and let $w \to (y(w), \tau(w), z(w))$ be a *direct* public sector mechanism such that

$$(y(w), \tau(w), z(w)) \in \Phi(w,C,H) \text{ for all } w \in W \text{ and}$$

$$\int_W (\tau(w) - c(z(w)))dP(w) \geq 0.$$

Thus the direct tax function $\tau(\cdot)$ finances public goods $z(\cdot)$ and

$$u(w, y(w), \tau(w), z(w)) = \max_{(y,\tau,z)\in C\times H} u(w, y, \tau, z) \text{ for all } w \in W.$$

Second, let $y \to C(y)$ be a set-valued mapping given by $C(y) = \{\tau \in T : (y, \tau) \in C\}$ and let $t(\cdot) : Y \to T$ be a $(B(Y), B(T))$-measurable function such that

$$t(y) \in C(y) \text{ for all } y \in Y \text{ and } t(y) = \min\{\tau : \tau \in C(y)\},$$

Since the set-valued mapping $y \to C(y)$ is $B(Y)$-measurable with nonempty closed values in Y, such a function exists (see Bertsekas and Shreve (1978), Proposition 7.33).[9] Moreover, since $t(y) \in C(y)$ for all $y \in Y$, $0 \leq t(y) \leq y$ for all $y \in Y$.

Claim 1: $(y(w), \tau(w), z(w)) = (y(w), t(y(w)), z(w))$ for all $w \in W$.

If not then for some agent type $w' \in W$, $\tau(w') \neq t(y(w'))$. Since

$$t(y(w)) = \min\{\tau : \tau \in C(y(w))\} \text{ for all } w \in W,$$

$\tau(w') \neq t(y(w'))$ implies that $\tau(w') > t(y(w'))$. But given [A-1](2), $\tau(w') > t(y(w'))$ contradicts the fact that

$$u(w, y(w), \tau(w), z(w)) = \max_{(y,\tau,z)\in C\times H} u(w, y, \tau, z)$$

for each w. Thus, $\tau(w) = t(y(w))$ for all $w \in W$, and thus,

$$\int_W (t(y(w)) - c(z(w)))dP(w) \geq 0.$$

[9]Given [A-1](2), $u(w, y, t(y), z) = \max_{\tau\in C(y)} u(w, y, \tau, z)$ for all $(w, y, z) \in W \times Y \times G$ where $t(\cdot)$ is any selection from $y \to C(y) = \{\tau : (y, \tau) \in C\}$, $C \in \Lambda$, such that $t(y) = \min\{\tau : \tau \in C(y)\}$.

Claim 2: For each $w \in W$,

$$u(w, y(w), t(y(w)), z(w)) \geq u(w, y, t(y), z(w')) \text{ for all } y \in Y \text{ and } w' \in W.$$

Suppose not. Then for some $w' \in W$, $y'' \in Y$, and $w'' \in W$,

$$\begin{aligned} u(w', y'', t(y''), z(w'')) &> u(w', y(w'), t(y(w')), z(w')) \\ &= u(w', y(w'), \tau(w'), z(w')). \end{aligned} \qquad (*)$$

Since $(y'', t(y''), z(w'')) \in C \times H$, (*) contradicts the fact that

$$u(w, y(w), \tau(w), z(w)) = \max_{(y,\tau,z) \in C \times H} u(w, y, \tau, z)$$

for each w. Thus, the $(y(\cdot), t(\cdot), z(\cdot))$ is contained in $\Psi \cap \Gamma \cap \Pi$ and

$$(y(w), t(y(w)), z(w)) \in \Phi(w, C, H) \text{ for all } w \in W,$$

so that

$$u(w, y(w), t(y(w)), z(w)) = \max_{(y,\tau,z) \in C \times H} u(w, y, \tau, z) \text{ for all } w \in W.$$

(2) Let $(y(\cdot), t(\cdot), z(\cdot)) \in \Psi \cap \Gamma \cap \Pi$ and let $C = \mathrm{cl}[Gr(t(\cdot))]$, where cl denotes closure and $Gr(t(\cdot))$ is the graph of the indirect tax function $t(\cdot)$. Thus,

$$Gr(t(\cdot)) = \{(y, \tau) \in Y \times T : \tau = t(y)\}.$$

Also, let $H = \{z\}$ where z is that public goods vector in G such that

$$z(w) = z \text{ for all } w \in W.$$

Thus, $H \in S$.

First note that since $t(\cdot)$ is defined on all of Y, $\mathrm{proj}_Y[\mathrm{cl}[Gr(t(\cdot))]] = Y$. Note also that since $0 \leq t(y) \leq y$ for all $y \in Y$, $0 \leq \tau \leq y$ for all $(y, \tau) \in \mathrm{cl}[Gr(t(\cdot))]$. Thus, $\mathrm{cl}[Gr(t(\cdot))] \in \Lambda$.

Second, since $C = \mathrm{cl}[\mathrm{Gr}(t(\cdot))]$ it is easy to see that

$$(y(w), t(y(w))) \in C \text{ for all } w \in W ,$$

and thus

$$u(w, y(w), t(y(w)), z(w)) \leq \max_{(y,\tau,z)\in C\times H} u(w, y, \tau, z) \text{ for all } w \in W .$$

Suppose now that for some agent type $w' \in W$ there is some 3-tuple $((y', \tau'), z') \in C \times H$ such that

$$u(w', y(w'), t(y(w')), z(w')) < u(w', y', \tau', z').$$

Since $H = \{z\}$, $u(w', y', \tau', z') = u(w', y', \tau', z(w')) = u(w', y', \tau', z)$. Moreover, since (y', τ') is in the closure of the graph of $t(\cdot)$ and since $u(w', \cdot, \cdot, z(w'))$ is continuous on $Y \times T$, there is an income and tax liability pair $(\overline{y}, \overline{\tau})$ contained in the graph of $t(\cdot)$ such that

$$u(w', y(w'), t(y(w')), z(w')) < u(w', \overline{y}, \overline{\tau}, z(w')).$$

Thus,

$$u(w', y(w'), t(y(w')), z(w')) < u(w', \overline{y}, t(\overline{y}), z(w'))$$

where $t(\overline{y}) = \overline{\tau}$. This contradicts the assumption that $(y(\cdot), t(\cdot), z(\cdot)) \in \Psi$ (i.e., the assumption that $(y(\cdot), t(\cdot), z(\cdot))$ is incentive compatible with $z(\cdot)$ a constant function). Thus, since $(y(w), t(y(w)), z(w)) \in C \times H$ for all $w \in W$ and since

$$u(w, y(w), t(y(w)), z(w)) = \max_{(y,\tau,z)\in C\times H} u(w, y, \tau, z) \text{ for all } w \in W,$$

we can conclude that $(y(w), t(y(w)), z(w)) \in \Phi(w, C, H)$ for all $w \in W$. Moreover, since $(y(\cdot), t(y(\cdot)), z(\cdot))$ is a measurable selection from $\Phi(\cdot, C, H)$, and since

$$\int_W (t(y(w)) - c(z(w))) dP(w) \geq 0,$$

we can conclude that $(C, H) = (\mathrm{cl}[\mathrm{Gr}(t(\cdot))], \{z\}) \in R$ Q.E.D.

4. The Existence of an Optimal Public Sector Mechanism

The μ-tax design problem with public goods (i.e., the public sector mechanism design problem) can be written compactly as

$$\max_{(y(\cdot),t(\cdot),z(\cdot))\in\Psi\cap\Gamma\cap\Pi} \int_W u(w,y(w),t(y(w)),z(w))d\mu(w) \tag{25}$$

The μ-menu design problem is given by

$$\max_{(C,H)\in R} \int_W u\hat{}(w,C,H)d\mu(w). \tag{26}$$

We now have our main result stating necessary and sufficient conditions for the existence of an optimal public sector mechanism. The proof of this Theorem follows directly from Theorem 1 and its proof.

Theorem 2

Suppose [A-1], [A-2], and [A-3] hold. Let μ be any finite measure equivalent to the probability measure P. Then the μ-tax design problem has a solution if and only if the μ-menu design problem has a solution. In particular, the following statements are true:

(1) If the public sector mechanism $(y(\cdot),t(\cdot),z(\cdot))\in\Psi\cap\Gamma\cap\Pi$

$$\text{maximizes } \int_W u(w,y(w),t(y(w)),z(w))d\mu(w) \text{ over } \Psi\cap\Gamma\cap\Pi,$$

then the pair of menus $(cl[Gr(t(\cdot))],\{z\})$, where $cl[Gr(t(\cdot))]$ is the closure of the graph of the indirect tax function $t(\cdot)$ and z is the public goods vector in G such that $z(w)=z$ for all $w\in W$, is contained in R and

$$\text{maximizes } \int_W u\hat{}(w,C,H)d\mu(w) \text{ over R.}$$

(2) If $(C,H)\in R$

$$\text{maximizes } \int_W u\hat{}(w,C,H)d\mu(w) \text{ over R,}$$

then the mechanism $(y(\cdot), t(\cdot), z(\cdot))$ constructed in (a) and (b) below is contained in $\Psi \cap \Gamma \cap \Pi$ and

$$\text{maximizes } \int_W u(w, y(w), t(y(w)), z(w))d\mu(w) \text{ over } \Psi \cap \Gamma \cap \Pi.$$

(a) $y(\cdot)$ is the direct income function and $z(\cdot)$ the direct public goods function corresponding to a direct public sector mechanism

$$(y(\cdot), \tau(\cdot), z(\cdot)) \in M(W, Y) \times M(W, T) \times M(W, G)$$

such that $(y(w), \tau(w), z(w)) \in \Phi(w, C, H)$ for all $w \in W$ and

$$\int_W (\tau(w) - c(z(w)))dP(w) \geq 0;$$

(b) $t(\cdot) : Y \rightarrow T$ is a $(B(Y), B(T))$-measurable function such that

$$t(y) \in C(y) \text{ for all } y \in Y \text{ and } t(y) = \min\{\tau : \tau \in C(y)\},$$

where $y \rightarrow C(y)$ is the set-valued mapping given by

$$C(y) = \{\tau \in T : (y, \tau) \in C\} \text{ for each } y \in Y.$$

Our next Theorem is our existence result for the menu design problem.

Theorem 3

Suppose [A-1], [A-2], and [A-3] hold. Then for each finite measure μ equivalent to the probability measure P, there exists a pair of menus $(C^*, H^*) \in R$ such that

$$\int_W u\hat{}(w, C^*, H^*)d\mu(w) = \max_{(C,H)\in R} \int_W u\hat{}(w, C, H)d\mu(w).$$

PROOF: Since $(C,H) \rightarrow u\hat{}(w,C,H)$ is upper semicontinuous on $\Lambda \times S$ for each w, $(C,H) \rightarrow \int_W u\hat{}(w,C,H)d\mu(w)$ is upper semicontinuous on $\Lambda \times S$ for each finite measure μ. This follows from Fatou's Lemma (e.g., see Dudley (1989)) and the definition of upper semicontinuity. Thus, since $R \subset \Lambda \times S$ is compact, the existence of an optimal pair of menus $(C^*, H^*) \in R$ follows from the classical Weierstrass Maximum Theorem. Q.E.D.

Now we have the main result of the paper. This result states that the general public sector mechanism design problem has a solution, and moreover, that this solution is efficient.

Theorem 4
Suppose [A-1], [A-2], and [A-3] hold. Then for each finite measure μ equivalent to the probability measure P, there exists a public sector mechanism

$$(y^*(\cdot), t^*(\cdot), z^*(\cdot)) \in \Psi \cap \Gamma \cap \Pi$$

such that

$$\int_W u(w, y^*(w), t^*(y^*(w)), z^*(w)) d\mu(w)$$

$$= \max_{(y(\cdot), t(\cdot), z(\cdot)) \in \Psi \cap \Gamma \cap \Pi} \int_W u(w, y(w), t(y(w)), z(w)) d\mu(w).$$

Moreover, the public sector mechanism $(y^*(\cdot), t^*(\cdot), z^*(\cdot)) \in \Psi \cap \Gamma \cap \Pi$ is efficient.

PROOF: By Theorem 3, for each P-equivalent finite measure μ there exists an optimal pair of menus $(C^*, H^*) \in R$.

By part (1) of Theorem 1 this implies that there exists a corresponding optimal public sector mechanism $(y^*(\cdot), t^*(\cdot), z^*(\cdot)) \in \Psi \cap \Gamma \cap \Pi$.

By Proposition 1 such a mechanism is efficient. Q.E.D.

5. Optimal Budget Balancing Public Sector Mechanisms

In this section, we identify a condition sufficient to guarantee that the optimal public sector mechanism can be chosen so as to generate *no* excess revenue (i.e., so that the optimal mechanism is budget balancing). The budget surplus problem is, of course, well-known in the public finance literature (e.g., see Groves and Loeb (1975), Groves and Ledyard (1977), and Green and Laffont (1977)).

We begin by considering the best response mapping

$$w \to \Phi(w, C, H),$$

corresponding to the menus (C,H). The closed set $\Phi(w,C,H)$ is the type w agent's set of optimal income, tax liability, and public goods 3-tuples given menus (C,H). Since for all $w \in W$ and all $(C,H) \in \Lambda \times S$

$$\Phi(w,C,H) \subset K \times G,$$

and since $K \times G$ is a compact subset of R_+^{k+2} (recall K is a compact subset of R_+^2 and G is a compact subset of R_+^k), the collection of best response mappings,

$$\{\Phi(\cdot,C,H) : (C,H) \in \Lambda \times S\},$$

is P-integrably bounded.[10]

Now consider the set-valued mapping

$$(C,H) \to \int_W \Phi(w,C,H)dP(w), \tag{27}$$

where

$$\int_W \Phi(w,C,H)dP(w)$$
$$=: \{ \int_W f(w)dP(w) : f(w) = (y(w),\tau(w),z(w)) \in \Phi(w,C,H) \ \forall \ w \in W\}. \tag{28}$$

[10]Thus, there is a P-integrable, point-valued function $g(\cdot) : W \to R^{k+2}$ such that for any menus $(C,H) \in \Lambda \times S$, $\|x\| \leq g(w)$ for all $x \in R^{k+2}$ and $w \in W$ such that $x \in \Phi(w,C,H)$.

Proposition 5

(1) For each $(C,H) \in \Lambda \times S$, $\int_W \Phi(w,C,H)dP(w)$ is a nonempty, compact subset of R^{k+2}. Moreover, if the probability space of agent types (W,Σ,P) is atomless, then

$$\int_W \Phi(w,C,H)dP(w) \text{ is convex.}$$[11]

(2) The mapping $(C,H) \to \int_W \Phi(w,C,H)dP(w)$ is upper semicontinuous on $\Lambda \times S$.

PROOF: (1) It is easy to see that $\int_W \Phi(w,C,H)dP(w)$ is nonempty and bounded. To show that $\int_W \Phi(w,C,H)dP(w)$ is closed consider a sequence $\{x_n\}_n$ in $\int_W \Phi(w,C,H)dP(w)$ converging to $x \in R^{k+2}$. Let $\{f_n(\cdot)\}_n$ be a corresponding sequence of measurable selections from $\Phi(\cdot,C,H)$ such that for each n, $x_n = \int_W f_n(w)dP(w)$. Thus, $\lim_n \int_W f_n(w)dP(w) = x$. It follows from Fatou's Lemma in several dimensions (e.g., see Page (1991)), that there exists a $(\Sigma, B(Y) \times B(T) \times B(G))$-measurable selection $f(\cdot)$ from the mapping

$$w \to Ls\{f_n(w)\}$$

such that

$$x = \int_W f(w)dP(w).$$

Since $\Phi(\cdot,C,H)$ is closed-valued, $Ls\{f_n(w)\} \subset \Phi(w,C,H)$ for all $w \in W$. Thus, $f(w) \in \Phi(w,C,H)$ for all $w \in W$, and thus

$$x \in \int_W \Phi(w,C,H)dP(w).$$

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[11] A subset $E \in \Sigma$ is an atom of the probability space (W,Σ,P) if $P(E) > 0$ and for all $F \in \Sigma$ such that $F \subset E$ either $P(F) = 0$ or $P(E - F) = 0$. The probability space (W,Σ,P) is atomless if it contains no atoms.

The convexity of $\int_W \Phi(w,C,H)dP(w)$ whenever (W,Σ,P) is atomless follows directly from a classical result due to Richter (see Hildenbrand (1974), Theorem 3, page 62).

(2) Let $\{(C_n,H_n)\}_n$ be a sequence in $\Lambda \times S$ converging to $(C,H) \in \Lambda \times S$. Also let $\{x_n\}_n$ be a sequence such that for each n

$$x_n \in \int_W \Phi(w,C_n,H_n)dP(w).$$

Corresponding to the sequence $\{x_n\}_n$ there is a sequence of $(\Sigma, B(Y) \times B(T) \times B(G))$-measurable functions $\{f_n(\cdot)\}_n$ such that for each n, $f_n(\cdot)$ is a selection from $\Phi(\cdot,C_n,H_n)$ and

$$x_n = \int_W f_n(w)dP(w).$$

Since $\{x_n\}_n$ is bounded, without loss of generality, we can assume that $\{x_n\}_n$ converges to some $x \in R^{k+2}$. Thus, $\lim_n \int_W f_n(w)dP(w) = x$. It follows from Fatou's Lemma in several dimensions, that there exists a $(\Sigma, B(Y) \times B(T) \times B(G))$-measurable selection $f(\cdot)$ from $w \to Ls\{f_n(w)\}$ such that

$$x = \int_W f(w)dP(w).$$

Since the sequence $\{f_n(\cdot)\}_n$ is uniformly bounded on W, for each $w \in W$ there is a subsequence $\{f_{n_k}(w)\}_k$ such that

$$f(w) = \lim_{n_k} f_{n_k}(w)$$

where $f_{n_k}(w) \in \Phi(w,C_{n_k},H_{n_k})$.

Since for each $w \in W$, $\Phi(w,\cdot,\cdot)$ is upper semicontinuous on $\Lambda \times S$, and since $(C_n, H_n) \to (C,H)$, we have for each $w \in W$ $f(w) \in \Phi(w,C,H)$. Thus, $x \in \int_W \Phi(w,C,H)dP(w)$ and we can conclude that $(C,H) \to \int_W \Phi(w,C,H)dP(w)$ is upper semicontinuous on $\Lambda \times S$ (see Theorem 1, p. 24 in Hildenbrand (1974)).

Q.E.D.

Our last Theorem identifies a condition sufficient to guarantee that an optimal public sector mechanism can be found that generates *no* excess revenue.

Theorem 5

Suppose [A-1], [A-2], and [A-3] hold, and let $(C^*,H^*) \in R$ be optimal menus. If $\int_W \Phi(w,C^*,H^*)dP(w)$ is convex then there exits a corresponding optimal public sector mechanism, $(y^*(\cdot),t^*(\cdot),z^*(\cdot)) \in \Psi \cap \Gamma \cap \Pi$, that generates no excess revenue. That is, there exists $(y^*(\cdot),t^*(\cdot),z^*(\cdot)) \in \Psi \cap \Gamma \cap \Pi$ such that

$$\int_W (t^*(y^*(w)) - c(z^*(w)))dP(w) = 0.$$

PROOF: Let $(y'(\cdot),\tau'(\cdot),z'(\cdot)) \in M(\Sigma,Y) \times M(\Sigma,T) \times M(\Sigma,G)$ be a *direct* mechanism such that

$$(y'(w),\tau'(w),z'(w)) \in \Phi(w,C^*,H^*) \text{ for all } w \in W,$$

and

$$\int_W (\tau'(w) - c(z'(w)))dP(w) > 0. \tag{29}$$

Thus, the direct mechanism $(y'(\cdot),\tau'(\cdot),z'(\cdot))$ generates excess revenue.

Since $H^* = \{z^*\}$ for some public goods vector $z^* \in G$ and since $z'(w) = z^*$ for all $w \in W$, (29) can be rewritten as

$$\int_W \tau'(w)dP(w) > c(z^*). \tag{30}$$

Now take the menu C^* and for each n form the menu $C_n{}^*$ by multiplying the tax liability corresponding to each income level by $(1-\frac{1}{n})$. Thus, each $(y_n, \tau_n) \in C_n{}^*$ is given by $(y,(1-\frac{1}{n})\tau)$ for some $(y,\tau) \in C^*$. Given assumption [A-1](2), for any n and any measurable selection $(y_n(\cdot), \tau_n(\cdot), z_n(\cdot))$ from $\Phi(\cdot, C_n{}^*, H^*)$, we have[12]

$$\begin{aligned} u(w, y_n(w), \tau_n(w), z_n(w)) &\geq u(w, y'(w), (1-\frac{1}{n})t'(w), z'(w)) \\ &> u(w, y'(w), t'(w), z'(w)). \end{aligned} \tag{31}$$

Thus, for any n and any measurable selection $(y_n(\cdot), \tau_n(\cdot), z_n(\cdot))$ from $\Phi(\cdot, C_n{}^*, H^*)$ it must be true that

$$\int_W \tau_n(w)dP(w) < c(z^*).$$

In particular, if for some n

$$\int_W \tau_n(w)dP(w) \geq c(z^*),$$

then it follows that $(C_n{}^*, H^*) \in R$. Given (31) this would contradict the optimality of (C^*, H^*).

Now observe that $\{(C_n{}^*, H^*)\}_n$ converges to (C^*, H^*). Let $\{x_n\}_n$ be a sequence such that for each n

$$x_n \in \int_W \Phi(w, C_n{}^*, H^*)dP(w).$$

Corresponding to the sequence $\{x_n\}_n$ there is a sequence of $(\Sigma, B(Y) \times B(T) \times B(G))$-measurable functions $\{f_n(\cdot)\}_n$ such that for each n, $f_n(\cdot) = (y_n(\cdot), \tau_n(\cdot), z_n(\cdot))$ is a measurable selection from $\Phi(\cdot, C_n{}^*, H^*)$ and

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[12]Note that for each n we have $z_n(w) = z^*$ for all $w \in W$.

$$x_n = \int_W f_n(w)dP(w) = \left(\int_W y_n(w)dP(w), \int_W \tau_n(w)dP(w), \int_W z_n(w)dP(w) \right);$$

For all n, we have

$$\int_W \tau_n(w)dP(w) < c(z^*). \tag{32}$$

Without loss of generality, assume that $\{x_n\}_n$ converges to some $x \in R^{k+2}$. Thus, $\lim_n \int_W f_n(w)dP(w) = x$. Again it follows from Fatou's Lemma in several dimensions and the upper semicontinuity of $\Phi(w,\cdot,\cdot)$ on $\Lambda \times S$ for each $w \in W$ that there exists a measurable selection $f(\cdot) = (y(\cdot), \tau(\cdot), z(\cdot))$ from $\Phi(\cdot, C^*, H^*)$ such that

$$x = \int_W f(w)dP(w) = \left(\int_W y(w)dP(w), \int_W \tau(w)dP(w), \int_W z(w)dP(w) \right).$$

From (32) it follows that

$$\int_W \tau(w)dP(w) \leq c(z^*). \tag{33}$$

Thus we have

$$x' = \left(\int_W y'(w)dP(w), \int_W \tau'(w)dP(w), \int_W z'(w)dP(w) \right) \in \int_W \Phi(w, C^*, H^*)dP(w)$$

with

$$\int_W \tau'(w)dP(w) > c(z^*),$$

and we have

$$x = \left(\int_W y(w)dP(w), \int_W \tau(w)dP(w), \int_W z(w)dP(w) \right) \in \int_W \Phi(w, C^*, H^*)dP(w)$$

with

$$\int_W \tau(w)dP(w) \leq c(z^*).$$

Given the convexity of $\int_W \Phi(w,C^*,H^*)dP(w)$, there exists therefore

$$x^* \in \int_W \Phi(w,C^*,H^*)dP(w),$$

and a corresponding measurable selection, $(y^*(\cdot),\tau^*(\cdot),z^*(\cdot))$, from $\Phi(\cdot,C^*,H^*)$ such that

$$\int_W \tau^*(w)dP(w) = c(z^*) .$$

Following the directions given in part (2) of Theorem 2 and using the direct public sector mechanism $(y^*(\cdot),\tau^*(\cdot),z^*(\cdot))$, we can construct an optimal public sector mechanism $(y^*(\cdot),t^*(\cdot),z^*(\cdot)) \in \Psi \cap \Gamma \cap \Pi$ such that

$$\int_W (t^*(y^*(w)) - c(z^*(w)))dP(w) = 0.$$

Q.E.D.

By part (1) of Proposition 5, $\int_W \Phi(w,C^*,H^*)dP(w)$ will be convex if the probability space of agent types is atomless. In addition $\int_W \Phi(w,C^*,H^*)dP(w)$ will be convex if the best response mapping $w \to \Phi(w,C^*,H^*)$ corresponding to the optimal menus (C^*,H^*) is single-valued.

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